AMERICAN JOURNAL of PHYSICS

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Citation: American Journal of Physics **36**, 420 (1968); doi: 10.1119/1.1974553
View online: http://dx.doi.org/10.1119/1.1974553
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/36/5?ver=pdfcov
Published by the American Association of Physics Teachers

From Eq. (2) we get

$$n \tan\beta \sec\beta d\beta = akdk/(k^2-h^2)^{1/2}$$

with h held constant. These two equations can be combined to give

$$\delta N = \frac{a}{\pi} kdk \frac{(k^2-h^2-n^2/a^2)^{1/2}}{k^2-h^2},$$

where δN is the number of lines of constant β cutting a line of constant h. The total number of roots, still for n fixed, is

$$\Delta N(n \text{ fixed}) = \delta N(2/\pi)dh$$
$$= \frac{La}{\pi^2} kdkdh \frac{(k^2-h^2-n^2/a^2)^{1/2}}{k^2-h^2}$$

The final result is arrived at by summing over the allowed values of n and multiplying by two to take into account the two linearly independent solutions $e^{\pm in\theta}$. Once again we replace the sum over n by an integral. To find the number of modes in the range dk it is somewhat simpler to perform the n integration first and then the h integral. If done in this order, the limits of integration are, for n, from 0 to $(k^2-h^2)^{1/2}a$, and for h, from 0 to k. This gives:

$$dN(\text{total}) = \frac{2La}{\pi^2} kdk \int_0^k dh \int_0^{(k^2-h^2)a^{1/2}} dn \times \frac{[k^2-h^2-(n/a)^2]^{1/2}}{k^2-h^2}$$
$$= \frac{2La}{\pi^2} kdk \int_0^k dh \tfrac{1}{4}(\pi a)$$
$$= (La^2/2\pi)k^2dk$$
$$= V_{\text{cyl}}(k^2dk/2\pi^2),$$

with $V_{\text{cyl}} = \pi a^2 L$. If the waves are electromagnetic, the effect is just to double the number of modes to take into account the two polarizations possible.

AMERICAN JOURNAL OF PHYSICS VOLUME 36, NUMBER 5 MAY 1968

Time Dependence of Quantum-State Amplitudes Demonstrated by Free Precession of Spins

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(Received 11 December 1967)

The time dependence of quantum-state amplitudes is calculated and experimentally demonstrated for the spin states of the rubidium atom. The mechanism of optical pumping is used to prepare the spin state oriented along the z axis. The magnetic field is then quickly switched from the z to the x direction and the light transmitted through the rubidium vapor is used as a probe to measure the probability that the atoms are in the various quantum states with respect to the z axis. The time dependence of absorbed light is calculated, first for the simplified case of spin $\frac{1}{2}$, then for spin 1, and finally for spin 2, which is the value observed in the experiment.

INTRODUCTION

The free precession of Rb atoms in a magnetic field is used to demonstate the time dependence of the probability for finding the atoms in a given quantum state. This demonstration was used as part of an introductory course in quantum mechanics given by Prof. R. Weiss at MIT. The texts for this course are notes by Kerman, Sartori, and Taylor[1] and Volume III of the Feynman lectures.[2] The notation and methods used for calculation are given in these two references.

[1] A. K. Kerman, L. Sartori, and E. F. Taylor, *Introduction to Quantum Physics* (The Education Research Center, MIT, unpublished).

[2] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley Publishing Co., Reading, Mass., 1965), Vol. 3.

Optical pumping is used to align the spins in the $+z$ direction (a good discussion of optical pumping is given by Bloom[3]). Initially the magnetic field is in the $+z$ direction. We quickly switch the field to the $+x$ direction and observe the light transmitted through a cell containing the aligned Rb vapor. We first treat the Rb atom as if it were a two-state system and calculate the time dependence of the transmitted light for spin $\frac{1}{2}$. We show that the transmitted light will be modulated at the classical precession frequency $\omega_p = \mu B/\mathcal{J}$. This calculation is then carried out for spin 1 and finally we consider the real Rb atom and calculate the behavior of a spin-2 system. For spin 1 and spin 2 the light is modulated at both ω_p and $2\omega_p$, simultaneously.

I. SPIN $\frac{1}{2}$

An analysis of a spin-$\frac{1}{2}$ system gives results which are similar to those which we observe with Rb atoms, and the mathematics is much simpler. We consider two states ($|+z\rangle$, $|-z\rangle$) for the ground level E_0 and two states for the excited state E_1. The energy-level diagram for this system is shown in Fig. 1. The separation between the states E_0 and E_1 corresponds to a wavelength of 7948 Å. A diagram of the apparatus is shown in Fig. 2. The rubidium lamp and circular polarizer produce left circularly polarized photons with an angular momentum along the z axis of $\mathcal{J}_z = +\hbar$. If we require the angular momentum along the z axis to be conserved, the only transitions to the E_1 level which occur are from the state (E_0, $|-z\rangle$) to the state (E_1, $|+z\rangle$). (See Fig. 1.). Since there are no E_1 states with $\mathcal{J}_z = \frac{3}{2}\hbar$, no transitions are made from the (E_0, $|+z\rangle$) state.

From the (E_1, $|+z\rangle$) state, the atoms may decay to either the (E_0, $|+z\rangle$) state or the (E_0, $|-z\rangle$) state since the photon produced may have $\mathcal{J}_2 = -\hbar$ or 0. If they decay to the $|-z\rangle$ state, however, they will absorb incident photons and go around the cycle again until finally most of the atoms are left in the (E_0, $|+z\rangle$) state. In this way we use optical pumping to produce the state $|+z\rangle$. We may measure the number of atoms in the $|-z\rangle$ state by measuring the light absorbed, since we know that absorption only occurs from the state $|-z\rangle$. If the atoms are in an arbitrary

[3] A. L. Bloom, Sci. Am. **203**, No. 4, 72 (Oct. 1960).

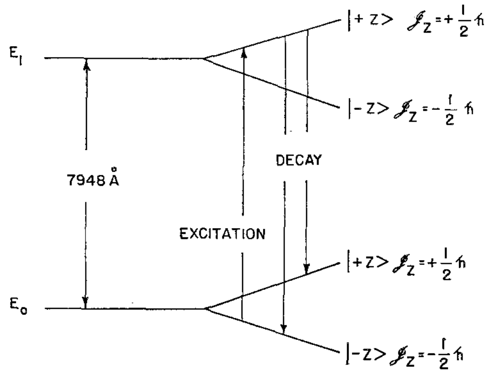

FIG. 1. Energy-level diagram for spin-$\frac{1}{2}$ system (splitting in magnetic field on right half of diagram is greatly exaggerated).

state $|\psi\rangle$, the light absorbed will be proportional to $|\langle -z|\psi\rangle|^2$, the probability of finding the atom in the state $|-z\rangle$.

An outline of the experiment is as follows:

(1) produce state $|+z\rangle$ by optical pumping with magnetic field in $+z$ direction;
(2) quickly switch field to $+x$ direction;
(3) measure transmitted light intensity with photocell to find $|\langle -z|\psi\rangle|^2$.

In the z representation we can write our initial state as

$$\begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_z.$$

This state in the x representation is

$$\begin{pmatrix} \langle +x|\psi\rangle \\ \langle -x|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle +x|z\rangle & \langle +x|-z\rangle \\ \langle -x|z\rangle & \langle -x|-z\rangle \end{pmatrix} \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}_x.$$

The effect of the magnetic field in the x direction will be to multiply the states $|x\rangle$, $|-x\rangle$ by the phase factor[4] $\exp(-iEt/\hbar)$. The energy of the

[4] See Ref. 2, Vol. 2, Chap. 34–3; A. Abragam, *The Principles of Nuclear Magnetism* (The Clarendon Press, Oxford, England, 1961), Chap. II.

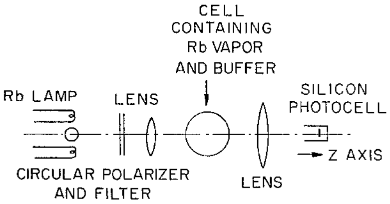

FIG. 2. Diagram of the apparatus.

states is $E=-\mu\cdot B$, so for the $|+x\rangle$ state $E_+=-\mu B$ and for the $|-x\rangle$ state $E_-=\mu B$. The classical precession frequency is $\omega_p=\mu B/\mathcal{J}$, where $\mathcal{J}$ is the angular momentum ($\frac{1}{2}\hbar$ in this case) so $E_+=-\hbar\omega_p/2$, $E_-=\hbar\omega_p/2$ and the phase factors are $\exp(i\omega_p t/2)$ for the $|+x\rangle$ state and $\exp(-i\omega_p t/2)$ for the $|-x\rangle$ state. In the x representation our quantum state is

$$\begin{pmatrix}\langle +x\mid\psi\rangle\\ \langle -x\mid\psi\rangle\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\exp(i\omega_p t/2)\\ -\exp(-i\omega_p t/2)\end{pmatrix}_x.$$

In order to transform back to the z representation, we use

$$\begin{pmatrix}\langle +z\mid\psi\rangle\\ \langle -z\mid\psi\rangle\end{pmatrix}=\begin{pmatrix}\langle +z\mid +x\rangle & \langle +z\mid -x\rangle\\ \langle -z\mid +x\rangle & \langle -z\mid -x\rangle\end{pmatrix}\begin{pmatrix}\langle +x\mid\psi\rangle\\ \langle -x\mid\psi\rangle\end{pmatrix};$$

and the state in the z representation is

$$\begin{pmatrix}\langle +z\mid\psi\rangle\\ \langle -z\mid\psi\rangle\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\exp(i\omega_p t/2)\\ -\exp(-i\omega_p t/2)\end{pmatrix}_x$$

$$=\begin{pmatrix}\cos\omega_p t/2\\ i\sin\omega_p t/2\end{pmatrix}_z.$$

The light absorbed is proportional to $|\langle -z\mid\psi\rangle|^2$ since this is the probability of finding an atom in the $|-z\rangle$ state, so the absorption is

$$A=C\,|\langle -z\mid\psi\rangle|^2=C\sin^2\omega_p t/2$$

$$=\tfrac{1}{2}C(1-\cos\omega_p t).$$

The transmitted light intensity is

$$T=I_0-A=I_0-C/2+(C/2)\cos\omega_p t,$$

where I_0 is the incident intensity and the transmitted intensity should be as shown in Fig. 3. Therefore, the current produced by the photocell should oscillate at the frequency ω_p. The experimental result for spin 2 is shown in Fig. 4. The signal was put through a high-pass filter to remove the initial step produced when the field is switched so the initial value is zero, but the oscillation is obviously present. We then vary the magnetic field to verify that the oscillation frequency is proportional to the magnetic field. There are three reasons why the oscillation amplitude decreases. First, the field is inhomogeneous so some spins are precessing at different frequencies than others. Second, the prepared state of aligned spins becomes disturbed after a large number of collisions with buffer gas atoms in the cell, and finally the measurement process of passing light through the cell produces transitions which re-orient the spins.

II. SPIN ONE

For spin one there are three states $|+1\rangle_z$, $|0\rangle_z$, $|-1\rangle_z$ with $\mathcal{J}_z=+\hbar$, 0, $-\hbar$, respectively. The initial state is prepared in the same way. Since the incident photons have $\mathcal{J}_z=+\hbar$, no transitions can take place from the $|+1\rangle_z$ state. Therefore, most of the atoms are "pumped" into the $|+1\rangle_z$ state. In the z representation the initial state is

$$\begin{pmatrix}\langle +1\mid\psi\rangle\\ \langle 0\mid\psi\rangle\\ \langle -1\mid\psi\rangle\end{pmatrix}_z=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}_z.$$

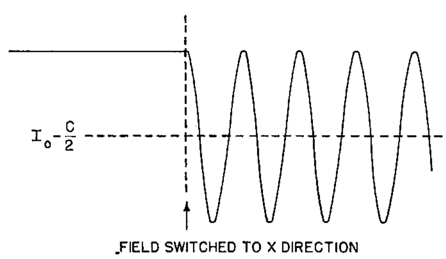

FIG 3. Calculated photocell current for spin $\frac{1}{2}$ as a function of time when the magnetic field is switched from the z to the x direction.

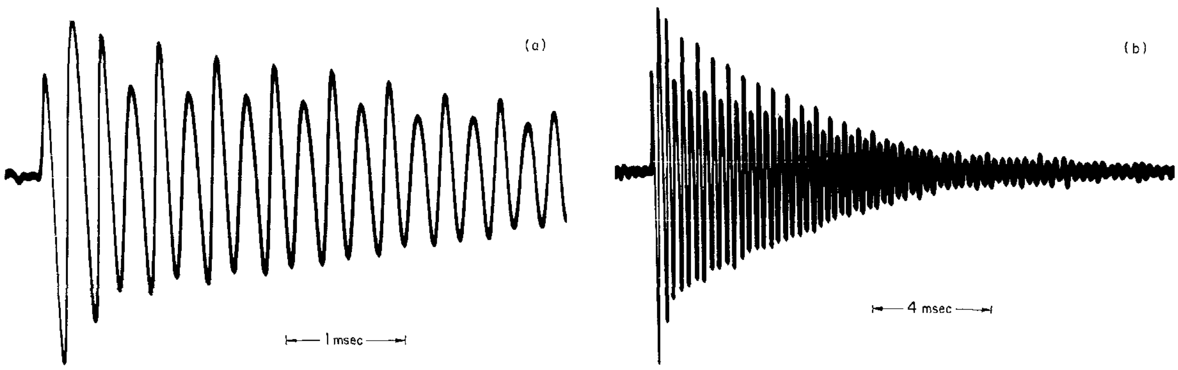

FIG. 4. Experimental photocell current as a function of time after the field is switched to the x direction. Frequency of $2\omega_p$ component is 4 kHz.

If we transform this into the x representation, we find

$$\begin{pmatrix} \langle +1 \mid \psi \rangle \\ \langle 0 \mid \psi \rangle \\ \langle -1 \mid \psi \rangle \end{pmatrix}_x = \tfrac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_z = \tfrac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}_x .$$

The transformation from z to x representation for spin one is given by Feynman.[5]

This time the phase factors $\exp(-iEt/\hbar)$ are $\exp(i\omega_p t)$ for the $|1\rangle_x$ state, 1 for the $|0\rangle_x$ state, and $\exp(-i\omega_p t)$ for the $|-1\rangle_x$ state. When the magnetic field is in the x direction the state is

$$\begin{pmatrix} \langle +1 \mid \psi \rangle \\ \langle 0 \mid \psi \rangle \\ \langle -1 \mid \psi \rangle \end{pmatrix}_x = \tfrac{1}{2} \begin{pmatrix} \exp(i\omega_p t) \\ -\sqrt{2} \\ \exp(-i\omega_p t) \end{pmatrix}_x .$$

When we transform this back to the z representation, we get

$$\begin{pmatrix} \langle +1 \mid \psi \rangle \\ \langle 0 \mid \psi \rangle \\ \langle -1 \mid \psi \rangle \end{pmatrix}_z = \tfrac{1}{4} \begin{pmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \exp(i\omega_p t) \\ -\sqrt{2} \\ \exp(-i\omega_p t) \end{pmatrix}_x = \begin{pmatrix} \tfrac{1}{2}\cos\omega_p t + \tfrac{1}{2} \\ (i/\sqrt{2})\sin\omega_p t \\ \tfrac{1}{2}\cos\omega_p t - \tfrac{1}{2} \end{pmatrix}_z .$$

Atoms in the state $|0\rangle_z$ or $|-1\rangle_z$ may absorb $\mathcal{J}_z = +\hbar$ photons. The transition probabilities for these processes are $|\langle +1 \mid J_+ \mid 0\rangle|^2$ and $|\langle 0 \mid J_+ \mid -1\rangle|^2$. These probabilities are equal for spin one, so the numbers of photons absorbed will just be proportional to $|\langle 0 \mid \psi\rangle|^2$ and $|\langle -1 \mid \psi\rangle|^2$ the relative numbers of atoms in these states.

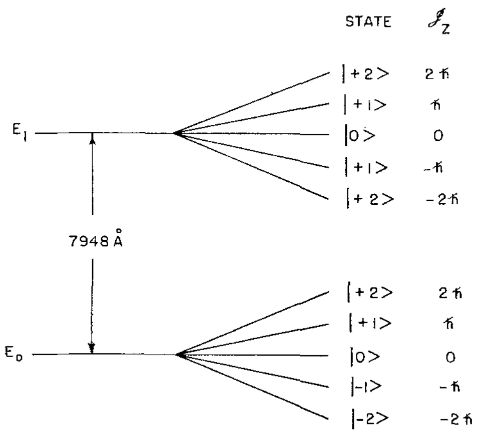

FIG. 5. Energy-level diagram for the states of ^{87}Rb which are important in this experiment.

[5] See Ref. 2, Vol. 3, Table 17-2.

The total absorption is

$$A=C\{|\langle 0\mid\psi\rangle|^2+|\langle -1\mid\psi\rangle|^2\}$$
$$=\tfrac{5}{8}C-\tfrac{1}{2}C\cos\omega_pt-\tfrac{1}{8}C\cos 2\omega_pt.$$

Here we have a term oscillating at the classical precession frequency ω_p and a term oscillating at $2\omega_p$.

III. SPIN TWO

Rubidium-87 has a nuclear spin $I=\frac{3}{2}$. This is coupled to the electron spin $s=\frac{1}{2}$ to form the states $F=2$ and $F=1$. The only state which cannot be excited by photons with $\mathcal{J}_z=\hbar$ is the state $F=2$, $|+2\rangle_z$ since the excited level E_1 contains no states $|+3\rangle_z$. (See Fig. 5.) So most of the atoms are "pumped" into the $F=2$, $|+2\rangle_z$ state; and we have a spin-two system.

We may calculate the transformation matrix from the z to the x representation using the methods outlined by Feynman.[6] The initial state in the x representation is then

$$\begin{pmatrix}\langle 2\mid\psi\rangle\\ \langle 1\mid\psi\rangle\\ \langle 0\mid\psi\rangle\\ \langle -1\mid\psi\rangle\\ \langle -2\mid\psi\rangle\end{pmatrix}_x=\begin{pmatrix}\frac14&\frac12&(\frac38)^{1/2}&\frac12&\frac14\\ -\frac12&-\frac12&0&\frac12&\frac12\\ (\frac38)^{1/2}&0&-\frac12&0&(\frac38)^{1/2}\\ -\frac12&\frac12&0&-\frac12&\frac12\\ \frac14&-\frac12&(\frac38)^{1/2}&-\frac12&\frac14\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}_z=\begin{pmatrix}\frac14\\ -\frac12\\ (\frac38)^{1/2}\\ -\frac12\\ \frac14\end{pmatrix}_x$$

The phase factors resulting from the magnetic field are $\exp(2i\omega_pt)$, $\exp(i\omega_pt)$, 1, $\exp(-i\omega_pt)$, $\exp(-2i\omega_pt)$ for the states $|2\rangle_x$, $|1\rangle_x$, $|0\rangle_x$, $|-1\rangle_x$, $|-2\rangle_x$.

The final state in the z representation is

$$\begin{pmatrix}\langle 2\mid\psi\rangle\\ \langle 1\mid\psi\rangle\\ \langle 0\mid\psi\rangle\\ \langle -1\mid\psi\rangle\\ \langle -2\mid\psi\rangle\end{pmatrix}_z=\begin{pmatrix}\frac14&-\frac12&(\frac38)^{1/2}&-\frac12&\frac14\\ \frac12&-\frac12&0&\frac12&-\frac12\\ (\frac38)^{1/2}&0&-\frac12&0&(\frac38)^{1/2}\\ \frac12&\frac12&0&-\frac12&-\frac12\\ \frac14&\frac12&(\frac38)^{1/2}&\frac12&\frac14\end{pmatrix}\begin{pmatrix}\frac14\exp(2i\omega_pt)\\ -\frac12\exp(i\omega_pt)\\ (\frac38)^{1/2}\\ -\frac12\exp(-i\omega_pt)\\ \frac14\exp(-2i\omega_pt)\end{pmatrix}_x$$

$$=\begin{pmatrix}\frac18\cos 2\omega_pt+\frac12\cos\omega_pt+\frac38\\ \frac14 i\sin 2\omega_pt+\frac12 i\sin\omega_pt\\ \frac12(\frac38)^{1/2}(\cos 2\omega_pt-1)\\ \frac14 i\sin 2\omega_pt-\frac12 i\sin\omega_pt\\ \frac18\cos 2\omega_pt-\frac12\cos\omega_pt+\frac38\end{pmatrix}_z$$

[6] See Ref. 2, Vol. 3, Chap. 18, Note 1.

The probability for an atom in the state $|M\rangle_z$ to absorb a photon with $\mathcal{J}_z=+\hbar$ is given by[7] $|\langle M+1 \mid J_+ \mid M\rangle|^2=J(J+1)-M(M+1)$. This is a spin-two system, so $J=2$. For the states $|+1\rangle$, $|0\rangle$, $|-1\rangle$, $|-2\rangle$, the respective transition probabilities are 4, 6, 6, 4. Therefore, the light absorbed is given by

$$A=C\{4\,|\langle 1 \mid \psi\rangle|^2+6\,|\langle 0 \mid \psi\rangle|^2+6\,|\langle -1 \mid \psi\rangle|^2 +4\,|\langle -2 \mid \psi\rangle|^2\}.$$

If we evaluate all of these terms we get,

$$\mathrm{A}=\tfrac{7}{2}C-2C\cos\omega_p t-\tfrac{3}{2}C\cos 2\omega_p t.$$

The transmitted light intensity is

$$T=I_0-A=I_0-\tfrac{7}{2}C+2C\cos\omega_p t+\tfrac{3}{2}C\cos 2\omega_p t,$$

and this is shown in Fig. 6. Again we get a term that oscillates at the classical precession frequency ω_p and another term which oscillates at $2\omega_p$. The calculated amplitude of the $2\omega_p$ component is 0.75 times the amplitude of the ω_p component. The experimental result is shown in Fig. 4. In this figure the amplitude of the $2\omega_p$ component is 7.5 times the amplitude of the ω_p component. This is partially due to the high-pass filter used to remove the initial transient.

IV. EXPERIMENTAL DETAILS

The Rb lamp is a $\frac{3}{4}$-in. diam bulb containing Rb and 1.6-mm Kr. It is excited by an rf oscillator at 100 MHz as described by Bell, Bloom, and Lynch.[8] The 7948-Å component of the light, which excites transitions to the $^2P_{1/2}$ level, is selected by a

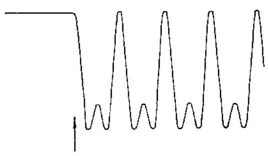

FIG. 6. Calculated photocell current for spin two as a function of time.

[7] A. Messiah, *Quantum Mechanics* (North-Holland Publishing Co., Amsterdam, 1962), Vol. 2, XIII. 19.

[8] W. E. Bell, A. L. Bloom, J. Lynch, Rev. Sci. Instr. **32**, 688 (1961).

Baird-Atomic interference filter. A Polaroid circular polarizer produces the left circularly polarized light used to excite the Rb atoms in the cell. The absorption cell is approximately 10 cm in diam and contains 66-mm Ne buffer gas. The typical operating temperature of the cell is 30°C which produces Rb vapor pressure of approximately 10^{-6} mm. The light transmitted through the cell is focused on an International Rectifier silicon photocell. The photocell current is amplified and passed through a filter with a passband from 0.7 ω_p to 25 kHz. This filter rejects the transient step in light intensity produced when the field is switched to the x direction. It also attenuates the ω_p signal by approximately 40% relative to the $2\omega_p$ signal.

The precession frequency ω_p for the results shown in Fig. 4 is 2 kHz corresponding to an x magnetic field of 0.003 G. The ω_p component and the 2 ω_p component are both present. The z field used to prepare the $|+z\rangle$ state was approximately 0.015 G. The earth's magnetic field was canceled by two pairs of Helmholz coils with 36-cm radius. The transient z field was produced by a pair of 10-cm-diam coils. These coils were made small to reduce the inductance so that the field could be switched more rapidly. In addition all large metal components were kept away from the region of the cell to prevent induced eddy currents from increasing the field switching time. The current in the coil producing the transient z field was switched at 10 Hz by a relay. In this experiment the main contribution to the decay time for the precession signal was field inhomogeneities.

A similar experiment was reported by Nagel and Haworth.[9] In their experiment the precession took place in the earth's field.

ACKNOWLEDGMENT

This experiment was suggested by Professor R. Weiss as a demonstration in an introductory course in quantum mechanics given by him at MIT.

[9] M. Nagel and F. E. Haworth, Am. J. Phys. **33**, 151 (1965).